YAMAHA

The Mathematics & History of Microtuning

DX7 IID·FD/D

DIGITAL PROGRAMMABLE ALGORITHM SYNTHESIZER

SUPPLEMENTAL BOOKLET

Welcome

Many people throughout history have noticed the relationship between music and mathematics. In the days of ancient Greece, music and mathematics were considered to be merely different aspects of the same discipline. In fact, most basic musical concepts such as intervals, scales, and tunings have been derived from mathematical and physical considerations.

This is the second in a series of booklets devoted to microtuning and its application on the DX7 II Digital Synthesizers.

Section 1 describes the relationship between music and mathematics and presents the fundamental tools of theoretical analysis.

Section 2 presents the story of this relationship in a historical context. The origins of scales represented by the preset tunings found in the DX7 II will be revealed. These presets are described in the previous booklet "Exploring the Preset Microtunings."

For continuing information concerning the DX7 II FD/D, consult AfterTouch, the official publication of the Yamaha Users Group. Many advanced functions will be discussed in its pages in the coming months. There will also be information regarding the availability of other materials concerning more advanced applications. To receive a free copy of AfterTouch every month, send your request to AfterTouch, P.O. Box 7938, Northridge, CA 91323-7938. On your letter or postcard, be sure to indicate that you are the owner of a DX7 II FD/D.

Contents

1 Section 1: Mathematics and Microtuning
2 Frequency
3 Intervals
3 Ratios
9 Cents
10 Anomalies

16 Section 2: History of Microtuning
16 History of Microtuning

27 Section 3: Microtuning Bibliography
28 Microtuning Bibliography

1 Mathematics and Microtuning

That musical concepts can be expressed with numbers has been recognized throughout human history. The historical development of this expression will be covered in Section 2. Before that journey into the past begins, some basic mathematical concepts will be presented. In preparation for this presentation, a concern common to many musicians must first be addressed.

In contemporary society, a phenomenon known as "math anxiety" is particularly common among artists of all types. This self-perceived lack of inherent mathematical ability is the result of deficiencies in our educational system, not in the intellectual capacity of artists. Many famous (and not so famous) scientists have also been accomplished musicians. There is no reason why musicians cannot also become mathematically adept. Read on, then, and fear not. A wondrous journey awaits.

Frequency

Sound is transmitted through the air by the action of molecules vibrating and colliding with each other. The source of the sound sets the nearby molecules into vibration which corresponds to the physical motion of the sound source itself. For example, imagine that a note is played on a DX7 II which is connected to an amplifier and a speaker. The speaker cone begins to vibrate, causing the air molecules nearby to vibrate in a corresponding fashion.

These nearby molecules collide with neighboring molecules, which collide with their neighbors, and so on until the air molecules next to our eardrums are set into motion. We perceive the sound when our eardrums are stimulated to vibrate in response to the motion of the nearby molecules.

The notes used to create most of the music we hear today are sounds for which the molecular vibrations described above are very regular. The sound source, air molecules, and eardrums all vibrate at a regular rate called the frequency, which is expressed in cycles per second. In honor of the contributions made to the study of acoustics by the German physicist Heinrich Hertz, cycles per second are also called hertz (abbreviated Hz).

Humans are capable of perceiving frequencies roughly between 20 Hz and 20,000 Hz. If our eardrums are stimulated to vibrate at a rate less than 20 Hz or greater than 20,000 Hz, we would not perceive the experience as a sound. In fact, depending on the amplitude or volume of the stimulation, we might not perceive it at all. This range of perception varies from person to person. The upper limit tends to decrease with advancing age and exposure to long periods of loud sounds.

The frequencies associated with specific musical notes depend mainly on historical factors. Before the advent of technological innovations such as tuning forks and electronic tuners, musical pitch varied with geography and history. During the Renaissance, the frequency associated with A above middle C was generally around 460 Hz. By the Baroque period, this A had dropped to roughly 415 Hz, almost a whole step lower. Since then, the frequency associated with this A slowly increased until it reached its current value of 440 Hz.

In order to specify the frequency being associated with a particular note, the following convention will be used thoughout this booklet. The name of the note will be followed immediately with the frequency of that note. For example, an A with a frequency of 440 Hz will be written A440.

Intervals

When two notes are played simultaneously, they are said to form an interval. An interval can also be defined as the relationship between the frequencies of any two notes. Intervals form the foundation from which scales and microtuning are generally studied.

The simplest interval other than the unison is the octave. Musically, two notes which form an octave share the same note name (for example, C). The notes sound almost identical, and yet one is higher than the other. The octave is one of the most compelling intervals because it demonstrates the cyclic, or repeating, nature of musical sound. Mathematically, an octave is obtained by doubling the frequency of any note. For example, the note which forms an octave with A440 is A880 (440 Hz $\times$ 2 = 880 Hz).

All intervals exhibit a subjective quality which manifests itself as the degree of consonance or dissonance with which they are perceived. Consonance is the degree to which an interval sounds pleasant or restful. A consonant interval has little or no musical tension or tendency to change. Such intervals are often found at the end of musical phrases or pieces. Dissonance is the degree to which an interval sounds unpleasant or rough. Dissonant intervals generally feel quite tense and unresolved. These intervals often precede consonant intervals in order to convey musical direction and movement. These perceptions are purely subjective and depend on the musical context in which they are found, but most people find general agreement about the consonance or dissonance of most intervals.

The octave is usually considered to be the most consonant interval. The other generally accepted consonant intervals are the perfect fifth, perfect fourth, major third, major sixth, minor third, and minor sixth. The intervals which are generally considered to be dissonant are the major second, minor seventh, minor second, major seventh, and the tritone (augmented fourth or diminished fifth). A mathematical basis for these subjective perceptions can be seen in the representation of intervals by ratios.

Ratios

The relationship between the frequencies of the two notes forming any interval can be described mathematically as a ratio. Numerically, ratios behave as fractions which are nothing more than one number being divided by another number. This means that you can determine the ratio formed by any two frequencies by simply dividing one frequency by the other.

Basics

Consider the frequencies 200 Hz and 100 Hz. Dividing 200 by 100 equals 2. In mathematical terms, 200/100 = 2. The frequencies 200 Hz and 100 Hz are said to be in the ratio of 2 to 1 (written 2:1 or 2/1). Any two numbers for which the result of dividing one by the other is 2 are said to be in the ratio of 2/1. For example, the ratios 10/5, 48/24, 1024/512, and 880/440 are all equivalent to the ratio 2/1 since the first number is twice the second number. This particular ratio describes the interval of an octave.

The advantage of using ratios to describe intervals is found in the fact that the specific frequencies which form an interval have no impact on the ratio which describes it. For example, consider the frequencies 200 Hz, 400 Hz, 500 Hz, and 1000 Hz. These frequencies represent two different octaves. The various ways to combine these frequencies are found in the following table.

Addition	400 + 200 = 600	1000 + 500 = 1500
Subtraction	400 - 200 = 200	1000 - 500 = 500
Multiplication	400 × 200 = 80,000	1000 × 500 = 500,000
Division (Ratio)	400 / 200 = 2/1	1000 / 500 = 2/1

As you can see, the result obtained by adding, subtracting, or multiplying the two frequencies together will depend on the frequencies themselves, even though both of the intervals are octaves. Only the ratio (division) provides the same result in both cases. Ratios therefore provide a consistent description of any interval without regard to the specific notes with which it is formed. This particular example illustrates that any pair of frequencies which form an octave will be in the ratio of 2/1.

Of course, other intervals are not described by the ratio 2/1. The ratios associated with intervals other than the octave have been derived using a variety of means throughout history. Much of this process is described in the next section of this booklet.

Consonance

One of the fundamental guiding principles which is evident throughout the development of musical mathematics is based in the study of psychoacoustics. This principle contends that intervals described by ratios of small whole numbers are more consonant and "harmonious" to the human ear than intervals described by ratios of large numbers or ratios of numbers other than whole numbers. The smaller the whole numbers in the ratio, the more consonant the interval. This is the objective mathematical concept which supports the subjective perception of consonance and dissonance described above.

Diatonic Intervals

With this in mind, here is a list of the diatonic intervals and the ratios which are generally accepted to describe them. This list also includes the decimal equivalent obtained by dividing the smaller number into the larger number of each ratio. This decimal equivalent will become important when equal temperament is considered in a mathematical context. Notice that the list is arranged roughly in order from most consonant to most dissonant.

Interval	Ratio	Decimal Equivalent
Unison	1/1	1
Octave	2/1	2
Perfect Fifth	3/2	1.5
Perfect Fourth	4/3	1.333333333...
Major Sixth	5/3	1.666666666...
Major Third	5/4	1.25
Minor Third	6/5	1.2
Minor Sixth	8/5	1.6
Major Second	9/8	1.125
Major Seventh	15/8	1.875
Minor Seventh	16/9	1.777777777...
Minor Second	16/15	1.066666666...
Tritone	45/32 or 64/45	1.40625 or 1.422222222...
(Diminished Fifth or Augmented Fourth)		

Prime Limits

If you examine the ratios listed above, you'll notice that none of the numbers in any of the ratios are multiples of a number higher than five. All of the numbers in these ratios are multiples of two, three, or five. These are examples of numbers known as primes. A prime is any number which can be divided evenly only by itself and one. Other primes include seven, eleven, and thirteen. It is most interesting that, while there is an infinity of primes, no one has yet derived a formula for generating them.

Musical theorists have limited the primes with which intervallic ratios are specified for various reasons throughout history. These reasons will be examined in the next section. For now, it is only important to realize that the pure intervals found in the traditional twelve tone diatonic scale are represented by ratios of numbers which are multiples of no prime higher than five.

This limitation excluding ratios of numbers which are multiples of primes higher than five is known as the "5-limit" (a term coined by microtonal composer Harry Partch in "Genesis of a Music"). It was adopted around 400 B.C. and has remained a foundation of scale development to this day. Many other scales can be generated by increasing this limiting number. For example, Partch developed a forty-three tone scale using intervals whose ratios consist of numbers which are multiples of primes no higher than eleven (the "11-limit").

Frequency

The ratios in the table above can be used to calculate the frequency of any note which forms a specific interval with another note of a known frequency. For example, to calculate the frequency of the E a perfect fifth above A440, multiply the known frequency by the value of the ratio (that is, by its decimal equivalent). In this case, 440 Hz × 1.5 = 660 Hz.

Addition & Subtraction

Intervals can be added together in order to form other intervals. For example, a perfect fourth and a perfect fifth placed back to back form an octave (C↑F + F↑C = C↑C). Interestingly, the same result is obtained by multiplying the ratios of the invervals being added. In the previous example, 4/3 × 3/2 = 12/6 = 2/1. This technique is very helpful when considering the effect of tuning several intervals upward one after the other. It will be used to illustrate various concepts throughout this and subsequent booklets.

A similar technique is used to subtract intervals. The ratio representing the interval to be subtracted is inverted (flipped over) and multiplied by the ratio describing the other interval. For example, subtracting a perfect fourth from a perfect fifth will result in a major second (C↑G - G↓D = C↑D). Using the technique described above, 3/2 × 3/4 = 9/8 which is the ratio of a major second. This technique is used to discern the effect of tuning intervals downward.

Individual Notes

Ratios can also be used to represent individual notes within the context of a key. For example, the note G could be represented by the ratio 3/2 in the key of C. This idea can be generalized to represent scale degrees with ratios in any key. For example, the third major scale degree would be represented by the ratio 5/4 while the third minor scale degree would be represented by 6/5. This notation is used to specify various scales without regard to a starting note.

The ratio of a note forming a specific interval with another note is calculated by multiplying the ratio of the known note by the ratio representing the interval. For example, the second degree of a major scale is represented by the ratio 9/8. To find the ratio of the note a perfect fourth above it, multiply the ratio by 4/3. Mathematically, 9/8 × 4/3 = 36/24 = 3/2. The note found a perfect fourth above the second degree of a major scale is the fifth degree. In the key of C, the note found a perfect fourth above D is G.

Similarly, the ratio describing the interval between any two notes can be calculated by inverting the ratio of the lower note and multiplying. For example, the second and fifth degrees of the major scale are represented by the ratios 9/8 and 3/2 respectively. After inverting the ratio of the lower note, the interval between them is calculated by multiplying the ratios together. Mathematically, 8/9 × 3/2 = 24/18 = 4/3. In other words, the interval formed by the second and fifth degrees of the major scale is a perfect fourth.

Octaves

As with frequencies, ratios describing individual notes or intervals which extend beyond one octave can be brought within the scope of the octave by dividing any such ratio by two. For example, two perfect fifths up from C leads to D one octave and one whole step higher (C↑G + G↑D). Mathematically, $3/2 \times 3/2 = 9/4$. To reduce this ratio by one octave, divide by two. This is equivalent to multiplying the ratio by 1/2. Mathematically, $9/4 \times 1/2 = 9/8$. This is the ratio which describes the major second, or whole step. Returning to the previous example, lowering the D by one octave would place it a major second above the original C. This technique of multiplying ratios by 1/2 in order to reduce them by one octave will be used throughout this and subsequent booklets.

Many Identical Intervals

Adding or subtracting a large number of intervals can become quite unwieldy. Fortunately, there is a shorthand method to express the addition or subtraction of a large number of identical intervals. This method involves the use of exponents. You may recall from high school math that multiplying a single number by itself several times can be expressed with exponents. For example, $2 \times 2 \times 2 = 2^3 = 8$. Exponents can also be applied to ratios. For example, adding five subsequent perfect fifths can be represented by the following formula.

$$3/2 \times 3/2 \times 3/2 \times 3/2 \times 3/2 = (3/2)^5 = 243/32$$

As this interval is larger than two octaves, multiply it by 1/2 twice in order to lower it by two octaves.

$$(3/2)^5 \times 1/2 \times 1/2 = (3/2)^5 \times (1/2)^2 = 243/32 \times 1/4 = 243/128$$

This ratio is not listed in the table of diatonic intervals above. It is close to the ratio for the major seventh (15/8). Tuning five perfect fifths upward and lowering by two octaves should result in exactly a major seventh. That it does not is one of the most puzzling aspects of microtuning. This puzzle is discussed below as the anomalies of microtuning are explained.

Equal Temperament

So far, only ratios of small whole numbers have been discussed. Ironically, the tuning system used exclusively today consists of no whole number ratios except the octave (2/1). As the first booklet in the microtuning series describes, the foundation of equal temperament lies in the division of the octave into twelve equal intervals called semitones which correspond to half steps. The process by which this division is accomplished involves exponents and their alter egos known as roots.

Suppose for a moment that the octave was to be divided into two exactly equal intervals. The decimal equivalent of the ratio describing these intervals will be represented by the letter "r" (for ratio). If two intervals of this ratio were to be added together, the resulting interval would be one octave. Mathematically, $r^2 = r \times r = 2$ (recall that the decimal equivalent of the ratio 2/1 is 2). The number which satisfies this equation cannot be represented by a whole number ratio. It cannot even be written as an exact decimal equivalent. This number is called the square root of two and is written $\sqrt{2}$. Its decimal equivalent is approximately 1.414213562. If you multiply this number by itself, you will find that the result will very nearly equal two.

Of course, the equal tempered scale divides the octave into twelve equal intervals called semitones. By adding twelve of these semitones together, the resulting interval will be one octave. If the letter r is used to represent the decimal equivalent of the ratio describing these semitones, the following formula illustrates this process.

$$r^{12} = r \times r \times r \times r \times r \times r \times r \times r \times r \times r \times r \times r = 2$$

Once again, the number which satisfies this equation cannot be expressed as a whole number ratio nor can it be written as an exact decimal equivalent. It is called the twelfth root of two and is written $\sqrt[12]{2}$. Its decimal equivalent is approximately 1.059463094. If you multiply this number times itself twelve times, you will find that the result will very nearly equal two.

The difference between pure minor seconds and equal tempered semitones can be demonstrated using decimal equivalents. Recall that a pure minor second is described by the ratio 16/15. Its decimal equivalent is approximately 1.066666667. As you have seen above, the ratio describing the equal tempered semitone is approximately 1.059463094. The nearest whole number ratio for this interval is 89/84. This indicates that the pure minor second is slightly wider than an equal tempered semitone.

Cents

In order to easily compare various intervals, each of the equally spaced semitones are further divided into 100 equal intervals called cents. This exceedingly small interval cannot be described exactly as a whole number ratio. Its decimal equivalent is approximately 1.00057779. The nearest whole number ratio is 1731/1730. Using this method of measurement, the intervals found in the equal tempered scale are easy to derive. They are listed in the following table along with their pure interval counterparts.

Interval	Equal Tempered Cents	Pure Cents
Unison	0	0
Minor Second	100	111.73
Major Second	200	203.91
Minor Third	300	315.64
Major Third	400	386.31
Perfect Fourth	500	498.04
Tritone	600	590.22 or 609.78
Perfect Fifth	700	701.95
Minor Sixth	800	813.69
Major Sixth	900	884.36
Minor Seventh	1000	996.09
Major Seventh	1100	1088.27
Octave	1200	1200

Formula

There is a formula for converting any ratio into its equivalent measure in cents. The formula involves the use of logarithms. While logarithms are related to exponents, it is not important that you fully understand them to use the formula. It merely requires that you have a calculator which calculates logarithms (abbreviated log). In the following formula, the letter "r" represents the decimal equivalent of the ratio you wish to convert and the letter "c" represents the number of cents into which the ratio will be converted.

$$c = 3986.313714 \times \log r$$

Here is the procedure for using this formula.

1. Determine the decimal equivalent of the ratio you wish to convert by dividing the upper number of the ratio by the lower number.
2. Calculate the log of this decimal equivalent using an appropriate calculator.
3. Multiply the number obtained in step 2 by 3986.313714.
4. The result of step 3 will be the number of cents in the selected ratio.

For example, this procedure will be used to find the number of cents in a pure perfect fifth.

1. A pure perfect fifth is represented by the ratio 3/2. The decimal equivalent of this ratio is 1.5.
2. The log of 1.5 is approximately 0.176091259. Mathematically, this is written $\log 1.5 = 0.176091259$.
3. Multiplying this number by 3986.313714 reveals the number of cents in a pure perfect fifth. Mathematically, $0.176091259 \times 3986.313714 = 701.9550008$ cents.

This indicates that the equal tempered perfect fifth (which is measured at exactly 700 cents) is almost two cents flatter than a pure perfect fifth. The other diatonic intervals can be similarly compared to reveal that, except for the octave, none of the intervals in the equal tempered scale are perfectly in tune. The reasons for this temperament's universal acceptance will be discussed in the next section of this booklet.

Anomalies

In the study of tuning and temperaments, one encounters small anomalies, or errors, which are inherent in the nature of intervals. Most of them were identified early in the history of musical theory. In general, an anomaly is the interval between the two notes found at the beginning and the end of certain series of pure intervals. This interval is usually quite small.

These anomalies illustrate a most curious fact about pure intervals. If pure intervals are repeatedly tuned upward or downward from a starting note, there will never be an exact recurrence (disregarding octave displacements) of any note in the sequence. Some of the notes so generated will be within one cent or less of other notes in the sequence, but there will be no exact duplications. The anomalies described below are the most common such differences.

Comma

The commas are the most common anomaly in the study of tuning and temperaments. They are encountered primarily while tuning pure perfect fifths. The specific intervals which give rise to each of the commas is described below.

Syntonic

The syntonic comma, also known as the comma of Didymus after its discoverer, becomes evident by tuning four perfect fifths upward followed by one major third downward. For example, tuning four perfect fifths up from C results in the notes C-G-D-A-E. A major third down from E returns to C. On a piano keyboard this C is exactly two octaves above the starting note of the sequence. However, if these intervals are in their pure form as described by the ratios listed earlier, the two Cs do not form perfect octaves. This can be demonstrated using the interval addition and subtraction techniques described above.

$$3/2 \times 3/2 \times 3/2 \times 3/2 \times 4/5 = 324/80$$

On the piano keyboard, this process results in a ratio of 4/1 which describes two notes exactly two octaves apart. Using pure intervals, this odd ratio of 384/80 is the result. You'll recall that multiplying a ratio by 1/2 reduces the interval it describes by one octave. This particular ratio must be lowered by two octaves in order to see the difference between the beginning and ending notes. This is accomplished by multiplying the ratio by 1/4.

$$324/80 \times 1/4 = 324/320 = 81/80$$

As you can see, the two Cs differ by a rather small but decidedly noticeable amount. This interval is the syntonic comma and it indicates that the note arrived at by tuning four perfect fifths upward followed by one major third and two octaves downward is 21.506 cents sharper than the original note.

Pythagorean

The Pythagorean comma becomes evident while generating a scale in the manner described by Pythagoras. By tuning twelve subsequent pure perfect fifths upward, all of the notes in the twelve tone scale will be generated. At the end of this process, the note normally considered to be the enharmonic equivalent of the starting note is reached. For example, tuning twelve pure perfect fifths upward from C will result in the note B#. In equal temperament, C and B# are enharmonic names for the same note. However, if this ending note is lowered by seven octaves, it becomes clear that they are not equivalent.

$$(3/2)^{12} \times (1/2)^7 = 531441/4096 \times 1/128 = 531441/524228$$

This rather cumbersome ratio demonstrates that B# is about 23.460 cents sharper than C in this system of tuning.

Note:

> *The equal tempered scale can be obtained by subtracting one twelfth of the Pythagorean comma from each of the pure fifths as they are tuned upward. The ratio of the pure fifth (3/2) encompasses 701.955 cents. One twelfth of the Pythagorean comma is 1.955 cents (23.460 × 1/12 = 1.955). Subtracting 1.955 cents from 701.955 cents leaves exactly 700 cents, the width of an equal tempered perfect fifth. Mathematically, 701.955 - 1.955 = 700.000. Tuning upward by fifths which are one twelfth of a Pythagorean comma narrower than pure fifths will result in the equal tempered twelve tone scale. This is the famous "circle of fifths" described below.*

Most musicians learn about the "circle of fifths" at some time during their musical education. In equal temperament, twelve consecutive perfect fifths close the circle at the starting note (enharmonically, since B# = C). Using pure fifths, the Pythagorean comma demonstrates that the circle of fifths is in fact a spiral. By continuing to add pure fifths from the thirteenth note, the spiral continues to expand but never exactly closes. By the time 41 pure fifths have been tuned, the resulting note is 19.8 cents below the starting note (disregarding octave displacements). The note at the end of 53 pure fifths is only 3.6 cents above the starting note. 306 pure fifths end up being only 1.8 cents below the starting note.

These and other "cycles" have been proposed at various times as alternative foundations upon which scales should be built. The next section of this booklet will place some of these proposals in historical perspective.

Great Diesis

On the equal tempered piano keyboard, three major thirds form exactly one octave. However, by tuning three pure major thirds upwards and lowering the result by one octave another anomaly appears.

$$(5/4)^3 \times 1/2 = 125/64 \times 1/2 = 125/128$$

This anomaly is known as the great diesis (pronounced di-a-sis) and indicates that three pure major thirds form an interval which is 41.059 cents flatter than a pure octave.

Schisma

The schisma (pronounced siz-ma or skiz-ma) is a very small anomaly which becomes apparent by tuning eight perfect fifths upward followed by one major third upward. The resulting note should be exactly five octaves above the starting note. The following formula illustrates this tuning process and lowers the resulting note by five octaves.

$$(3/2)^8 \times 5/4 \times (1/2)^5 = 6561/256 \times 5/4 \times 1/32 = 32805/32768$$

The schisma indicates that the note resulting from this tuning procedure is 1.954 cents sharper than the starting note.

Note:

> *It is quite interesting that the Pythagorean comma is equal to the syntonic comma plus the schisma (21.506 cents + 1.954 cents = 23.460 cents).*

Diaschisma

Tuning four perfect fifths downward followed by two major thirds downward will result in the enharmonic equivalent of the starting note three octaves lower. Raising this note by three octaves reveals another anomaly known as the diaschisma.

$$(2/3)^4 \times (4/5)^2 \times (2/1)^3 = 16/81 \times 16/25 \times 8/1 = 2048/2025$$

The diaschisma indicates that the note resulting from this tuning procedure is 19.553 cents sharper than the starting note.

2

History of Microtuning

Music has been an vital part of human activity throughout history. The development of formal musical structures such as scales and tunings can be traced back almost three thousand years. Of the three great civilizations to have flourished so long ago - Babylonia, Egypt, and China - records survive from the Chinese culture only. This, then, is the starting point of our journey into the past to discover the foundations of tunings and temperaments.

History of Microtuning

Ling Lun

The mathematical derivation of the pentatonic or five tone scale is attributed to Ling Lun, who was purported to be a court musician under Emperor Huang-Ti in the twenty-seventh century B.C. (although many modern scholars believe this antiquity to be exaggerated). He started with a length of bamboo called a "lu" which was closed at one end and open at the other. A tone is produced by blowing across the open end in a manner similar to that used play a tone with a bottle today.

The bamboo tube was measured into 81 equal parts. Another tube was cut to a length of 54 parts, which is two thirds of the original tube's length. Still another tube was cut to a length of 72 parts, which is the length of the second tube plus one third of that length. A fourth tube was cut to a length of 48 parts, which is two thirds of the previous tube's length. Finally, a fifth tube was cut to a length of 64 parts, which is derived by increasing the length of the fourth tube by one third again. This process results in a set of tubes which produce a series of pitches in the following ratios with respect to the frequency of the longest tube.

1/1		9/8		81/64		3/2		27/16		2/1
	9/8		9/8		32/27		9/8		32/27	

The sixth tube was exactly half the length of the longest tube. The ratios appearing below those of the tubes themselves represent the intervals between the consecutive notes produced by the tubes.

As you'll recall, the ratios 81/64 and 27/16 do not appear in the table of generally accepted pure ratios found in the previous section of this booklet. This is due to the fact that this scale, produced by altering the lengths of bamboo tubes by one third, is based on the "3-limit" (notice that none of the ratios contain numbers which are multiples of any prime higher than 3). It would be many centuries before these ratios would be replaced with smaller number ratios from the "5-limit" used in the table of pure intervals found in the previous section.

The intervals between consecutive notes in this scale can be preserved using pitches from the table of pure intervals by starting on the note which forms a minor seventh with the root of the scale.

16/9		1/1		9/8		4/3		3/2		16/9
	9/8		9/8		32/27		9/8		32/27	

By continuing the process described above, Ling Lun produced twelve lu which formed the first known twelve tone scale. It is believed that this scale was not used musically. The twelve lu were divided into two groups of six. The first group produced the following scale. A lu one half the length of the longest lu was added to provide the octave.

1/1 9/8 81/64 729/512 6561/4096 59049/32768 2/1
9/8 9/8 9/8 9/8 9/8 65539/59049

The second group produced essentially the same scale offset from the first group by roughly one half step.

2187/2048 19683/16384 177147/131072 3/2 27/16 243/128
9/8 9/8 65539/59049 9/8 9/8

Notice that each set forms a whole tone scale (consecutive intervals of 9/8 or 203.9 cents) with the exception of one interval with the ratio 65536/59049 (180.4 cents). This smaller interval occurs naturally in the process of constructing the lu. If the interval between 59049/32768 and 2/1 were adjusted to be 9/8, the octave would be sharp by a Pythagorean comma. Consecutive pitches between the two groups are separated by one of two intervals, 256/243 (90.2 cents) or 2187/2048 (113.7 cents).

Pythagoras

Pythagoras of Samos lived in Greece during the sixth century B.C. His prodigious studies in many areas of science have had a profound influence on Western thought to this day. This influence is so great that his name has become synonymous with many fundamental concepts.

Musically, Pythagoras took a similar approach to that of Ling Lun. Instead of bamboo tubes, however, he used a single string stretched between two bridges and held with a certain tension. This simple instrument was called a monochord. Pythagoras determined that the frequency of the pitch produced when the whole string was vibrating could be doubled by stopping the string at its midpoint. Of course, this produced a note one octave above the fundamental pitch. Successive octaves could be obtained by dividing the string into halves, quarters, eighths, and so on.

A scale consisting entirely of octaves is not very musically useful, so Pythagoras began dividing the string of the monochord into thirds. He found that setting two thirds of the string into vibration, a pitch was produced which formed an interval of 3/2 with the fundamental pitch of the whole string. During this process, Pythagoras also noticed that the interval formed by this pitch and the octave above the fundamental pitch of the monochord was 4/3. This process of forming intervals with the frequencies obtained from different proportional lengths of a single string is known as the Harmonic Proportion.

That Pythagoras did not proceed by dividing the string of the monochord into fifths, sevenths, and so on is a strange twist of humanistic fate that was to have an impact on musical theory to this day. Many historians feel that it was the perception of the number "3" as perfect or divine that prevented explorations into higher number ratios. Pythagoras and his followers established a brotherhood dedicated to a pure life and pure fifths based on the ratio 3/2. This idea spread throughout Greece and later to the rest of the known world.

As it so often happens in the history of music, practice preceded theory. The scales already in use could now be described using consecutive intervals of 3/2. For example, the notes produced by the ancient eight string lyre tuned in the Dorian mode could be expressed with the following ratios (arranged in descending order):

2/1 16/9 128/81 3/2 4/3 32/27 256/243 1/1

As consecutive intervals of 3/2, these pitches can be expressed in the following descending order:

3/2 1/1 4/3 16/9 32/27 128/81 256/243

The ratios used above to represent descending fifths have been adjusted to express the relative pitch of each note within its own octave. The scale thus derived remained the basis for tuning through the Middle Ages.

The sequence described above can be approximated on a modern keyboard by the notes B, E, A, D, G, C, F. Placed within a single ascending octave, this sequence becomes E, F, G, A, B, C, D. Oddly enough, this sequence is what we now call the Phrygian mode. The names of the modes were confused during the Middle Ages. Of course, these notes tuned in equal temperament are not those described by Pythagoras and used by the ancient Greeks. The DX7 II can be retuned to the ratios listed above. This will be covered in a subsequent booklet.

Archytas

While Pythagoras had made great progress quantifying the musical resources of his time, the scales he described were virtually unsingable unless the singers were accompanied by instruments so tuned. Archytas (c. 400 B.C.), a native of Tarentum, Italy, and a friend of Plato, substituted the ratio 5/4 for the Pythagorean 81/64 first used by Ling Lun. He also substituted the ratio 8/7 for 9/8. These actions opened the door to the eventual acceptance of ratios within the 5-limit and 7-limit as valid musical intervals.

Aristoxenus

A school of musical theorists known as the Harmonists developed between the time of Pythagoras and Archytas. In a reaction against the mathematical foundation of this school, Aristoxenus (c. 330 B.C.), a student of Aristotle, wrote as many as 453 works. Among them, "Elements of Harmony" is said to be the earliest extant treatise on Greek music. He believed that the ear, not mathematical calculation, should be the final judge of musical merit. Although he did not realize it, this thesis is consistent with the notion that small number ratios are inherently more consonant than large number ratios.

In this work, Aristoxenus also describes a process whereby whole tones (major seconds) are divided into halves (semitones), thirds (third tones), and quarters (quarter tones). He further rejects the musical application of any smaller intervals. This is the first reference to these small intervals.

Eratosthenes

Eratosthenes (276 - 196 B.C.), a native of Cyrene (Africa), was the director of the great library at Alexandria. It was he who substituted the ratio 6/5 for the Pythagorean 32/27, which affirmed Archytas' use of ratios within the 5-limit. In addition, he was the first proponent of the Arithmetical Proportion, although its discovery is generally attributed to Pythagoras. In this process, the string of the monochord is divided into a number of equal parts. The notes which result as the string is stopped at the various divisions are used to form a scale. Different scales can be constructed by dividing the string into a different number of parts. The Arithmetic Proportion would continue to be used by many musical theorists throughout history.

King Fang

During the third century B.C. a remarkable discovery was made by King Fang. He calculated the lengths and resulting frequencies for sixty lu that would result in a scale based on fifty-nine consecutive intervals of 3/2. He noticed that the frequency of the fifty-fourth lu was almost identical to the first lu (3.6 cents higher) after the octaves were taken into account. This anticipates the discovery of the "fifty-three cycle" of pure perfect fifths in the West by eighteen centuries.

Ptolemy

Ptolemy (139 A.D. - ?), a native of Alexandria, was a mathematician, astronomer, geographer, and musical theorist. His "Harmonics" may be the first complete exposition of just intonation in which he transforms the Greek scales into ratios of the smallest numbers compatible with the nature of each. In so doing, Ptolemy defined the scale which was to become the major scale in Europe.

1/1 9/8 5/4 4/3 3/2 5/3 15/8 2/1
9/8 10/9 16/15 9/8 10/9 9/8 16/15

This so-called Ptolemaic Sequence is only one of the many scales devised by this exceptional theorist.

There is evidence that just intonation was also being used in China as early as the third century B.C. A bronze kin, known as the "scholar's lute," was tuned to the following scale:

1/1 8/7 6/5 5/4 4/3 3/2 5/3 2/1
8/7 1/20 25/24 16/15 9/8 10/9 6/5

Ho Tcheng-Tien

Once again, the Chinese found themselves far ahead of the West in discoveries of a musical nature. Ho Tcheng-Tien (c. 370 - 447) gave the string lengths for the twelve tone equal tempered scale thirteen centuries before such a scale would be considered in Europe. Apparently, however, these string lengths were arrived at more by ear than by calculation. The formulation of equal temperament would be achieved virtually simultaneously in China and Europe some thirteen hundred years hence.

Walter Odington

During the Medieval period, an English monk by the name of Walter Odington (c. 1240 - 1280) noticed that it had become popular to sing intervals which were closer than 2/1 or 3/2. He wrote that some of the intervals in this new popular art (faux bourdon) were "imperfect consonances." In particular, Odington identified the thirds 5/4 and 6/5 as such imperfect consonances and stated that singers used them intuitively rather than the Pythagorean ratios 81/64 and 32/27. He also mentioned the major chord, possibly for the first time in recorded musical history.

Nicholas Faber & The Halberstadt Organ

On February 23, 1361, Nicholas Faber completed the construction of an organ for the cathedral in the Saxon city of Halberstadt. This organ had three manuals, the third of which consisted of nine front keys and five raised rear keys in groups of two and three. Excluding the two outer front keys, this was the first appearance of what was to become the modern keyboard. Its practical application to the musical developments of the time would soon follow.

The introduction of the now familiar keyboard was the result of the growing acceptance of thirds and fifths as simultaneous consonances. The beauty of Ptolemy's just intonation was effectively illustrated by the newly emerging triad. Ironically, the appearance of the keyboard was also a portent of the eventual rejection of just intonation as music became more harmonic and chromatic. The pure intervals required tempering in order to render this new music playable on keyboard instruments. This is due in large part to the fact that each note was now playing several different musical roles. The freedom of intonation inherent in the voice does not exist for the keyboard.

Francisco de Salinas

Among the first theorists to temper the pure intervals for the sake of the keyboard was a blind Spanish organist and professor by the name of Francisco de Salinas (1513 - 1590). He is generally credited with devising the meantone temperament, in which the perfect fifths are tuned slightly flat so that the major thirds can be tuned closer to pure. This also resulted in a major second or whole step which fell between the two whole step ratios 9/8 and 10/9. This average, or mean whole tone is the source of this temperament's name.

Due to the nature of the mean tone temperament, the twenty four possible major and minor triads fell into two groups: "good" and "bad." The sixteen good triads (eight major and eight minor) sounded much more pure than they do in equal temperament, but the remaining eight triads sounded much worse than they do today. Mean tone temperament had not solved the problem of playing in any key.

Don Nicola Vincentino

It was becoming clear that strictly pure intervals were not compatible with the keyboards then being developed. One solution was to redesign the keyboard so that many more than twelve notes per octave were available. One such instrument, called the Archicembalo, was built by Don Nicola Vincentino (c. 1550). This harpsichord like instrument included thirty one notes per octave arranged in six banks of keys. This is one of the first examples of alternative solutions to the problem of keyboards and their inherent limitations. Unfortunately for Vincentino, he enjoyed no support from his contemporaries.

Chu Tsai-yu

Although China had not developed harmonic music with anything near the vigor found in Europe at this time, there was a significant theoretical development made by Prince Chu Tsai-yu in 1596. He published a work in which he very accurately calculated the string lengths for the twelve tone equal tempered scale.

This discovery was the result of Prince Chu's puzzlement over the discrepancy between the just intonation of the "scholar's lute" and the Pythagorean tuning of the twelve lu. He resolved the discrepancy by devising the formula for equal temperament. As you'll recall, this formula divides the octave into twelve equal intervals. The ratio formed by consecutive notes in this system is $\sqrt[12]{2}$ /1 or approximately 1.059463094/1.

Marin Mersenne

Marin Mersenne (1588 - 1648) was a French monk, mathematician, and physicist. He was one of the first theorists to advocate the use of ratios within the 7-limit. He noticed that the natural overtone series of the trumpet produced a major triad and other, more complex harmonies. He decided that, since the natural harmonic series went beyond the major tonality, so should the musical resources of the time go beyond the arbitrarily imposed 5-limit. He was the first to declare that the interval 7/6 was consonant and designed many keyboards with greater resources than the already common 7-white-5-black keyboard.

He is also generally considered to be the first European to correctly identify the formula for equal temperament. However, it would not be until 1688, forty years after his death, that the first organ would be tuned according to his formula by Art Schnitger in Hamburg.

Andreas Werckmeister

The now familiar keyboard had become widely accepted by the Baroque period. The virtues of equal temperament were being extolled by some and eschewed by others. In an attempt to reach a compromise between pure sounding intervals and harmonic capability, many musicians and theorists devised various temperaments with which music in any key could be played but which also closely approximated as many pure intervals as possible. These are often referred to as the "well temperaments" because they worked well in any key.

It was these well temperaments which inspired J.S. Bach to write the "Well Tempered Clavier." Contrary to popular belief, this set of keyboard pieces written in each of the major and minor keys was not intended for performance in equal temperament. It was well known at the time that well temperaments afforded each key its own "color" or unique character. This was due to the fact that the semitones were not identical as they are in equal temperament. It was Bach's intention to illustrate these "key colors" in his "Well Tempered Clavier."

Andreas Werckmeister (1645 - 1706) was an organist, composer, and theorist highly respected by Handel, Buxtehude, and many other musicians. He was one of the first to clearly state the principles of well temperament. Among these principles was the premise that well temperaments should favor the primary intervals found in keys with few sharps or flats at the expense of tonalities with many sharps or flats. This was a very practical idea since most of the music being composed at that time was written in these simple key signatures due to the long established use of key sensitive tunings such as the mean tone temperament. Later, even though composers such as Mozart and Haydn had developed the compositional facility to modulate into any key, they tended to use key signatures with few sharps or flats because of the widespread use of well temperaments. Music performed on keyboards so tuned sounded better in C major and other simple keys.

Johann Philipp Kirnberger

Johann Philipp Kirnberger (1721 - 1783) was a student of J.S. Bach. As a composer, conductor, and theorist, he developed many well temperaments. One of these temperaments is found in the DX7 II permanent microtuning memory.

Francescantonio Vallotti & Thomas Young

Many of the well temperaments were based on adjusting some or all of the fifths in a Pythagorean tuning of consecutive perfect fifths. You'll recall that by tuning upward from a given note by twelve intervals of 3/2, the final note will form an interval with the starting note which is sharper than an octave by 23.46 cents (the Pythagorean comma). Many well temperaments are based on tuning the octave pure and placing the comma elsewhere (hopefully where it will not be played). One solution was to divide the comma into two, three, four, six, or even twelve parts and temper certain intervals by the amount represented by the partial comma.

Francescantonio Vallotti and Thomas Young independently devised a well temperament in which the first six Pythagorean fifths were lowered by one sixth of a comma while the second six fifths were untempered. The only difference between these temperaments was that Vallotti started on F and Young started on C. This temperament is also found in the permanent memory of the DX7 II.

Hermann Helmholtz & Alexander Ellis

As seemingly impossible to stem as the advancing tide, equal temperament finally achieved universal acceptance by the late nineteenth century. As the Romantic and later periods in music history saw more chromatic modulations and extended chords, total harmonic flexibility was required of keyboard instruments.

Hermann Helmholtz (1821 - 1894) compiled what has since been considered the definitive exposition of all major acoustical, theoretical, and pertinent physiological knowledge gathered to his time. "On the Sensations of Tone" is still considered one of the essential texts on the subject to this day. He and his English translator, Alexander Ellis (1814 - 1890), have been instrumental in recording and preserving the various fundamental musical concepts which have been developed throughout history.

It is interesting to note that Helmholtz was actively opposed to equal temperament. He felt that it was not appropriate to sacrifice pure intervals for the convenience of keyboard instruments.

As for Ellis, his major contribution to the study of musical theory was the invention of musical cents. As you'll recall, there are 1200 cents per octave, or 100 cents per equal tempered semitone. This single development has allowed all theorists since that time to compare the size of intervals with relative ease.

Continued Interest in Pure Intervals

Although equal temperament had become the standard tuning, there were several stalwart individuals who continued to experiment with other scales and tuning systems on keyboard instruments primarily of their own design and construction. For the most part, they felt that equal temperament was an unacceptable compromise and set about designing keyboards which overcame its limitations.

Perronet Thompson, British general and member of Parliment, built an organ with forty tones per octave. He also wrote a book entitled "On the Principles and Practice of Just Intonation" in 1866. In the latter part of the nineteenth century, Henry Ward Poole constructed a cardboard model of a keyboard with 100 keys per octave. At about the same time James Paul White built three harmoniums with keyboards designed to play the "fifty three cycle" of pure fifths. R.H.M. Bosanquet designed and built an instrument he called the Enharmonic Harmonium which was based on fifty three equally tempered degrees per octave. He even tried to start a business enterprise in which customers could order organs with different numbers of keys per octave.

Colin Brown, a lecturer on music at Andersonian University in Glasgow, devised another solution which was not based on Pythagorean fifths. His "Voice Harmonium" incorporated more than forty tones per octave and had a total range of five octaves. It was designed to play fifteen different major scales and triads as well as fifteen minor scales and triads in pure just intonation. It was not Brown's intention to build an instrument for the performance of future music, but rather for the just performance of existing musical material.

Harry Partch

The pioneer of just intonation in this century was Harry Partch (1901 - 1974). His development of a scale consisting of forty-three tones per octave is based on small number just ratios and stands as a milestone in music history. He designed and built an entire orchestra of acoustic instruments to play his music. There are several recordings of his work currently available on the Columbia label. The musical world is indeed fortunate that Partch also recorded his theories in a voluminous work entitled "Genesis of a Music." The Bibliography includes a complete reference to this work.

Microtuning in Electronic Music

What of the use of electronic instruments with alternate scales and tuning systems? Aside from some experimental work conceived in the hallowed halls of academia, practically all commercially available synthesizers have been immutably tuned in equal temperament. It is ironic that, with the popular advent of musical electronics, the door to alternate tunings should have been easily opened. Instead, the inertia which resists change has prevented the implementation of that which should be trivial for modern microprocessor-based instruments.

Fortunately, this situation has recently changed forever. Yamaha has broken the grip of exclusivity enjoyed by equal temperament for over 150 years. The DX7 II Digital Synthesizers provide full control over the tuning of each note they produce. For the first time, widely available electronic instruments present the opportunity to expand the musical horizon in a fundamental way.

The first booklet in this series, "Exploring the Preset Microtunings," provides an introduction to the DX7 II's microtonal capability. Subsequent booklets will include information about creating your own microtunings and using them in your music. These booklets serve as a springboard into vast new musical worlds awaiting exploration.

3 Microtuning Bibliography

Microtuning Bibliography

The following works provide additional information regarding microtuning, scales, and historical temperaments.

Genesis of a Music
Written by Harry Partch
Published by Da Capo Press
227 West 17th St.
New York, NY 10011

On the Sensations of Tone
Written by Hermann Helmholtz
Published by Dover Publications
180 Varick St.
New York, NY 10014

Fundamentals of Musical Acoustics
Written by Arthur H. Benade
Published by Oxford University Press
NY

Tuning and Temperament
Written by J. Murray Barbour
Published by Michigan State College Press
East Lansing, MI

Intervals, Scales, and Temperaments
Written by L.S. Lloyd & Hugh Boyle
Published by St. Martin's Press
NY

The Equal Beating Temperaments
Written by Owen Jorgenson
Published by The Sunbury Press

Tuning the Historical Temperaments by Ear
Written by Owen Jorgenson
Published by Northern Michigan University Press
Marquette, MI

Harpsichord Tuning
Written by G.C. Klop
Distributed in the USA by The Sunbury Press

Articles "Interval" and "Temperaments"
Appearing in The New Grove Dictionary of Music and Musicians
Written by Mark Lindley
Published by MacMillan Publishers
London, England

The Seventh Dragon
Written by Anita T. Sullivan
Published by Metamorphous Press
P.O. Box 1712
Lake Oswego, OR 97034

Notes